

Twisted Dirac operators and dynamical zeta functions

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Abstract. In this paper, we consider the dynamical zeta functions of Ruelle and Selberg associated with the geodesic flow of a compact hyperbolic odd dimensional manifold X . These functions are initially defined on one complex variable s in some right half-plane of \mathbb{C} . Our goal is the continue meromorphically the dynamical zeta functions to the whole complex plane, using the Selberg trace formula for arbitrary, not necessarily unitary, representations χ of the fundamental group. First, we prove a trace formula for the integral operator $D_\chi^\sharp(\sigma)e^{-t(D_\chi^\sharp(\sigma))^2}$, induced by the twisted Dirac operator $D_\chi^\sharp(\sigma)$ on X . Then we use these results to establish the meromorphic continuation of the dynamical zeta functions to \mathbb{C} .

Keywords: Twisted Dirac operator, Ruelle zeta function, Selberg zeta function.

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1 Introduction

The zeta functions of Ruelle and Selberg are dynamical functions associated with the geodesic flow of a compact hyperbolic manifold X . They can be represented by infinite products, which involve exponentials of the lengths of the prime closed geodesics on X .

Our geometrical setting consists of compact hyperbolic manifolds X of odd dimension d , obtained as $X := \Gamma \backslash \mathbb{H}^d$, where \mathbb{H}^d denotes the d -dimensional real hyperbolic space and Γ a discrete torsion-free subgroup of $\mathrm{SO}^0(d, 1)$. Here, we identify \mathbb{H}^d with $\mathrm{SO}^0(d, 1)/\mathrm{SO}(d)$. We set $G := \mathrm{SO}^0(d, 1)$ and $K := \mathrm{SO}(d)$. Let $G = KAN$ be the Iwasawa decomposition of G and M the centralizer of A in K . Let \widehat{M} be the set of the equivalent classes of unitary irreducible representations of M .

We denote by $[\gamma]$ the Γ -conjugacy class of $\gamma \in \Gamma$. One can associate to $[\gamma] \neq e$, a unique closed geodesic c_γ . We consider the lengths $l(\gamma)$ of the corresponding closed geodesics c_γ . We call the conjugacy class $[\gamma]$ prime if there exist no $k \in \mathbb{N}$ with $k > 1$ and $\gamma_0 \in \Gamma$ such that $\gamma = \gamma_0^k$. The prime geodesics can be viewed as the geodesics of minimum length.

Let $\sigma \in \widehat{M}$. We give here the definition of the zeta functions in terms of finite dimensional representations χ of Γ . The twisted Selberg zeta function $Z(s; \sigma, \chi)$ is defined by

$$Z(s; \sigma, \chi) := \prod_{\substack{[\gamma] \neq e, \\ [\gamma] \text{ prime}}} \prod_{k=0}^{\infty} \det \left(\mathrm{Id} - (\chi(\gamma) \otimes \sigma(m_\gamma) \otimes S^k(\mathrm{Ad}(m_\gamma a_\gamma)_{\bar{\mathfrak{n}}})) e^{-(s+|\rho|)l(\gamma)} \right),$$

where $\bar{\mathfrak{n}}$ is the sum of the negative root spaces of the system $(\mathfrak{g}, \mathfrak{a})$ and $S^k(\mathrm{Ad}(m_\gamma a_\gamma)_{\bar{\mathfrak{n}}})$ denotes the k -th symmetric power of the adjoint map $\mathrm{Ad}(m_\gamma a_\gamma)$ restricted to $\bar{\mathfrak{n}}$. $Z(s; \sigma, \chi)$ is defined for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > c_1$, for some constant $c_1 > 0$.

We define also the twisted Ruelle zeta function $R(s; \sigma, \chi)$ by

$$R(s; \sigma, \chi) := \prod_{\substack{[\gamma] \neq e \\ [\gamma] \text{ prime}}} \det(\mathrm{Id} - \chi(\gamma) \otimes \sigma(m_\gamma) e^{-sl(\gamma)})^{(-1)^{d-1}},$$

where $s \in \mathbb{C}$ with $\mathrm{Re}(s) > c_2$, for some constant $c_2 > 0$.

The meromorphic continuation of the Ruelle and Selberg zeta function has been long studied by Fried ([Fri86]), Bunke and Olbrich ([BO95]), Wotzke ([Wot08]) and Pfaff ([Pfa12]), but only for specific representations of the subgroup Γ . Under certain assumptions for the representation of Γ i.e., the representation is orthogonal and

acyclic, Fried ([Fri86, Theorem 1]) managed to prove that the Ruelle zeta function admits a meromorphic continuation to the whole complex plane and moreover that is regular at zero and its absolute value at zero is equal to the Ray-Singer analytic torsion. Bunke and Olbrich ([BO95]) dealt with the meromorphic continuation of the dynamical zeta functions but in the case of a unitary representation of Γ . Wotzke ([Wot08]) generalized in his thesis the results of Fried to the case of the induced representations of Γ , i.e., representations arising from restrictions of finite dimensional representations of $G = \mathrm{SO}^0(d, 1)$. Pfaff ([Pfa12]) extended these results to the co-finite case using the induced representation as well.

In [Spi15], the existing results for the meromorphic continuation of the dynamical zeta function are generalized to the case of an arbitrary representation of Γ , not necessarily unitary. In this paper, we study further this case, under the condition that the action of the restricted Weyl group W_A on \widehat{M} is not trivial, i.e., $w\sigma \neq \sigma$, where w is a non-trivial element of W_A . We have then to introduce three more zeta functions the symmetrized $S(s, \sigma, \chi)$, super $Z^s(s, \sigma, \chi)$ and super Ruelle $R^s(s, \sigma, \chi)$ zeta function (see Definitions 3.3, 3.4 and 3.5). To be able to prove the meromorphic continuation of the zeta functions, we introduce the twisted Dirac operator, defined in a similar way as the twisted Bochner-Laplace operator in [Mül11].

We give here a short description of the twisted Dirac operator. As before, we consider an arbitrary finite dimensional representation χ of Γ . Let E_χ be the flat vector bundle over X . Let \widehat{K} be the set of the equivalent classes of unitary irreducible representations of K . Let s be the spin representation of K and $\tau(\sigma) \in \widehat{K}$. We define the representation $\tau_s(\sigma)$ of K by $\tau_s(\sigma) := s \otimes \tau(\sigma)$. We consider then the locally homogeneous vector bundle $E_{\tau_s(\sigma)}$ over X . We let $D(\sigma)$ be the Dirac operator associated with $\tau_s(\sigma)$. Let $D_\chi^\sharp(\sigma)$ be the twisted Dirac operator acting on $C^\infty(X, E_{\tau_s(\sigma)} \otimes E_\chi)$. Locally, it is given by the following formula.

$$\widetilde{D}_\chi^\sharp(\sigma) = \widetilde{D}(\sigma) \otimes \mathrm{Id}_{V_\chi},$$

where $\widetilde{D}_\chi^\sharp(\sigma)$ and $\widetilde{D}(\sigma)$ are the lifts to \widetilde{X} of $D_\chi^\sharp(\sigma)$ and $D(\sigma)$, respectively. Since the representation of Γ is not necessarily unitary, we deal with non self-adjoint operators. Nevertheless, one can observe by the formula above that the twisted Dirac operator $D_\chi^\sharp(\sigma)$ has the same principal symbol as $D(\sigma)$. Hence, it has nice spectral properties, i.e., its spectrum is a discrete subset of a positive cone in \mathbb{C} . Let $D_\chi^\sharp(\sigma)e^{-tD_\chi^\sharp(\sigma)}$ be the associated operator acting on the space of smooth sections of the vector bundle $E_{\tau_s(\sigma)} \otimes E_\chi$. By [Mül11, Lemma 2.4], this is an integral operator with smooth kernel and hence of trace class. We derive a corresponding trace formula for $D_\chi^\sharp(\sigma)e^{-tD_\chi^\sharp(\sigma)}$.

Theorem 1.1. *For every $\sigma \in \widehat{M}$,*

$$\mathrm{Tr}(D_\chi^\sharp(\sigma)e^{-t(D_\chi^\sharp(\sigma))^2}) = \sum_{[\gamma] \neq e} \frac{-2\pi i}{(4\pi t)^{3/2}} \frac{l^2(\gamma) \mathrm{tr}(\chi(\gamma) \otimes (\sigma(m_\gamma) - w\sigma(m_\gamma)))}{n_\Gamma(\gamma)D(\gamma)} e^{-l^2(\gamma)/4t}.$$

This trace formula is the key point to prove the meromorphic continuation of the super zeta function.

Theorem 1.2. *The super zeta function $Z^s(s; \sigma, \chi)$ admits a meromorphic continuation to the whole complex plane \mathbb{C} . The singularities are located at $\{s_k^\pm = \pm i\lambda_k : \lambda_k \in \mathrm{spec}(D_\chi^\sharp(\sigma)), k \in \mathbb{N}\}$ of order $\pm m_s(\lambda_k)$, where $m_s(\lambda_k) = m(\lambda_k) - m(-\lambda_k) \in \mathbb{N}$ and $m(\lambda_k)$ denotes the algebraic multiplicity of the eigenvalue λ_k .*

In [Spi15], the differential operator $A_\chi^\sharp(\sigma)$, which is induced by the twisted Bochner-Laplace operator $\Delta_{r,\chi}^\sharp$, is introduced. Moreover, a trace formula for the corresponding heat semi-group $e^{-tA_\chi^\sharp(\sigma)}$ is proved. In the present case, where $w\sigma \neq \sigma$, we derive a trace formula for the operator $e^{-tA_\chi^\sharp(\sigma)}$, which is slightly different.

Theorem 1.3. *For every $\sigma \in \widehat{M}$ we have*

$$\begin{aligned} \mathrm{Tr}(e^{-tA_\chi^\sharp(\sigma)}) &= 2 \dim(V_\chi) \mathrm{Vol}(X) \int_{\mathbb{R}} e^{-t\lambda^2} P_\sigma(i\lambda) d\lambda \\ &\quad + \sum_{[\gamma] \neq e} \frac{l(\gamma)}{n_\Gamma(\gamma)} L_{\mathrm{sym}}(\gamma; \sigma + w\sigma) \frac{e^{-l(\gamma)^2/4t}}{(4\pi t)^{1/2}}, \end{aligned}$$

where

$$L_{\mathrm{sym}}(\gamma; \sigma) = \frac{\mathrm{tr}(\sigma(m_\gamma) \otimes \chi(\gamma)) e^{-|\rho|l(\gamma)}}{\det(\mathrm{Id} - \mathrm{Ad}(m_\gamma a_\gamma)_{\bar{\mathfrak{n}}})}.$$

The trace formula in Theorem 1.3 is again the tool which we use to prove the meromorphic continuation of the symmetrized zeta function.

Theorem 1.4. *The symmetrized zeta function $S(s; \sigma, \chi)$ admits a meromorphic continuation to the whole complex plane \mathbb{C} . The set of the singularities equals $\{s_k^\pm = \pm i\sqrt{\mu_k} : \mu_k \in \mathrm{spec}(A_\chi^\sharp(\sigma)), k \in \mathbb{N}\}$. The orders of the singularities are equal to $m(\mu_k)$, where $m(\mu_k) \in \mathbb{N}$ denotes the algebraic multiplicity of the eigenvalue μ_k . For $\mu_0 = 0$, the order of the singularity s_0 is equal to $2m(0)$.*

Finally, we prove the following theorems.

Theorem 1.5. *The Selberg zeta function $Z(s; \sigma, \chi)$ admits a meromorphic continuation to the whole complex plane \mathbb{C} . The set of the singularities equals to $\{s_k^\pm = \pm i\lambda_k : \lambda_k \in \text{spec}(D_\chi^\sharp(\sigma)), k \in \mathbb{N}\}$. The orders of the singularities are equal to $\frac{1}{2}(\pm m_s(\lambda_k) + m(\lambda_k^2))$. For $\lambda_0 = 0$, the order of the singularity is equal to $m(0)$.*

Theorem 1.6. *For every $\sigma \in \widehat{M}$, the Ruelle zeta function $R(s; \sigma, \chi)$ admits a meromorphic continuation to the whole complex plane \mathbb{C} .*

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2 Preliminaries

Let X be a compact hyperbolic locally symmetric manifold with universal covering the real hyperbolic space \mathbb{H}^d , where $d = 2n + 1$ is an odd integer. We let $G := \text{Spin}(d, 1)$, $K := \text{Spin}(d)$ be the universal coverings of $\text{SO}^0(d, 1)$ and $\text{SO}(d)$, respectively. We put $\widetilde{X} := G/K$. We consider the Lie algebras $\mathfrak{g}, \mathfrak{k}$ of G and K , respectively. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of \mathfrak{g} . Let \mathfrak{a} be a Cartan subalgebra of \mathfrak{p} . Then, there exists a canonical isomorphism $T_{eK} \cong \mathfrak{p}$. We consider the subgroup A of G with Lie algebra \mathfrak{a} . We let $M := \text{Centr}_K(A)$ be the centralizer of A in K . Then, $M = \text{Spin}(d-1)$ or $M = \text{SO}(d-1)$. Let \mathfrak{m} be its Lie algebra and \mathfrak{b} a Cartan subalgebra of \mathfrak{m} . Let \mathfrak{h} be a Cartan subalgebra of \mathfrak{g} . Let $\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C}, \mathfrak{m}_\mathbb{C}$ be the complexifications of $\mathfrak{g}, \mathfrak{h}$ and \mathfrak{m} , respectively. We consider the inner product

$$\langle Y_1, Y_2 \rangle_0 := \frac{1}{2(d-1)} B(Y_1, Y_2), \quad Y_1, Y_2 \in \mathfrak{g}, \quad (2.1)$$

induced by the Killing form $B(Y_1, Y_2) := \text{Tr}(\text{ad}(Y_1) \circ \text{ad}(Y_2))$ on $\mathfrak{g} \times \mathfrak{g}$. The restriction of $\langle \cdot, \cdot \rangle_0$ to \mathfrak{p} defines an inner product on \mathfrak{p} and hence induces a G -invariant riemannian metric on \widetilde{X} , which has constant curvature -1 . Then, \widetilde{X} , equipped with this metric, is isometric to \mathbb{H}^d . Let $\Gamma \subset G$ be a discrete cocompact torsion free subgroup of G . Then, $X := \Gamma \backslash \widetilde{X}$ is a locally symmetric compact hyperbolic manifold of dimension d .

Let $G = KAN$ be the standard Iwasawa decomposition of G . Let $\Delta^+(\mathfrak{g}, \mathfrak{a})$ be the set of positive roots of the system $(\mathfrak{g}, \mathfrak{a})$. Then, $\Delta^+(\mathfrak{g}, \mathfrak{a}) = \{\alpha\}$. Let $H_\mathbb{R} \in \mathfrak{a}$ such that $\alpha(H_\mathbb{R}) = 1$. We set

$$A^+ := \{\exp(tH_\mathbb{R}) : t \in \mathbb{R}^+\}, \quad (2.2)$$

$$\rho := \frac{1}{2} \sum_{\alpha \in \Delta^+(\mathfrak{g}, \mathfrak{a})} \dim(\mathfrak{g}_\alpha) \alpha, \quad (2.3)$$

$$\rho_{\mathfrak{m}} := \frac{1}{2} \sum_{\alpha \in \Delta^+(\mathfrak{m}_{\mathbb{C}}, \mathfrak{b})} \alpha. \quad (2.4)$$

Let \widehat{K}, \widehat{M} be the sets of equivalent classes of irreducible unitary representations of K and M , respectively. Let ν_τ, ν_σ be the highest weights of $\tau \in \widehat{K}$ and $\sigma \in \widehat{M}$, respectively. Then, by [BO95, p. 20]

$$\nu_\tau = (\nu_1, \dots, \nu_n),$$

where $\nu_1 \geq \dots \geq \nu_n$ and $\nu_i, i = 1, \dots, n$ are all integers or all half integers and

$$\nu_\sigma = (\nu_1, \dots, \nu_{n-1}, \nu_n), \quad (2.5)$$

where $\nu_1 \geq \dots \geq \nu_{n-1} \geq |\nu_n|$ and $\nu_i, i = 1, \dots, n$ are all integers or all half integers .

Let (s, S) be the spin representation of K and (s^\pm, S^\pm) the half spin representations of M as in [Spi15, p. 7]. By [BO95, p. 20], the highest weight ν_s of s is given by $(\frac{1}{2}, \dots, \frac{1}{2})$, and the highest weights ν_{s^\pm} of s^\pm are $\nu_{s^\pm} = (\frac{1}{2}, \dots, \pm \frac{1}{2})$.

3 Twisted Selberg and Ruelle zeta functions

The twisted Ruelle and Selberg zeta functions are associated with the geodesic flow on the sphere vector bundle $S(X) = \Gamma \backslash G/M$ of X . By ([GKM68]), there is a 1-1 correspondence between the closed geodesics on a manifold X with negative sectional curvature and the non-trivial conjugacy classes of Γ . Since Γ is a cocompact subgroup of G , we realize every element $\gamma \in \Gamma - \{e\}$ as hyperbolic. By [Wal76, Lemma 6.5], there exist a $g \in G$, a $m_\gamma \in M$, and an $a_\gamma \in A^+$, such that $g^{-1}\gamma g = m_\gamma a_\gamma$. The element m_γ is determined up to conjugacy classes in M , and the element a_γ depends only on γ .

Let c_γ be the closed geodesic on X , associated with the conjugacy class $[\gamma]$. We denote by $l(\gamma)$ the length of c_γ . An element $\gamma \in \Gamma$ is primitive, if there exists no $n \in \mathbb{N}$ with $n > 1$ and $\gamma_0 \in \Gamma$ such that $\gamma = \gamma_0^n$. The prime geodesics on X are those geodesics whose periodic orbits are of minimal length. They are associated with the primitive elements $\gamma_0 \in \Gamma$.

Let now $\chi: \Gamma \rightarrow \text{GL}(V_\chi)$ be an arbitrary finite dimensional representation of Γ and $\sigma \in \widehat{M}$.

Definition 3.1. The twisted Selberg zeta function $Z(s; \sigma, \chi)$ for X is defined by the infinite product

$$Z(s; \sigma, \chi) := \prod_{\substack{[\gamma] \neq e \\ [\gamma] \text{ prime}}} \prod_{k=0}^{\infty} \det \left(\text{Id} - (\chi(\gamma) \otimes \sigma(m_\gamma) \otimes S^k(\text{Ad}(m_\gamma a_\gamma)|_{\bar{\mathfrak{n}}})) e^{-(s+|\rho|)l(\gamma)} \right), \quad (3.1)$$

where $s \in \mathbb{C}$, $\bar{\mathfrak{n}} = \theta \mathfrak{n}$ is the sum of the negative root spaces of \mathfrak{a} , $S^k(\text{Ad}(m_\gamma a_\gamma)|_{\bar{\mathfrak{n}}})$ denotes the k -th symmetric power of the adjoint map $\text{Ad}(m_\gamma a_\gamma)$ restricted to $\bar{\mathfrak{n}}$, and ρ is as in (2.3).

By [Spi15, Proposition 3.4], there exists a positive constant c , such that the infinite product in (3.1) converges absolutely and uniformly on compact subsets of the half-plane $\text{Re}(s) > c$.

Definition 3.2. The twisted Ruelle zeta function $R(s; \sigma, \chi)$ for X is defined by the infinite product

$$R(s; \sigma, \chi) := \prod_{\substack{[\gamma] \neq e \\ [\gamma] \text{ prime}}} \det \left(\text{Id} - \chi(\gamma) \otimes \sigma(m_\gamma) e^{-sl(\gamma)} \right)^{(-1)^{d-1}}. \quad (3.2)$$

By [Spi15, Proposition 3.5], there exists a positive constant r , such that the infinite product in (3.2) converges absolutely and uniformly on compact subsets of the half-plane $\operatorname{Re}(s) > r$.

Let $M' = \operatorname{Norm}_K(A)$ be the normalizer of A in K . We define the restricted Weyl group by $W_A := M'/M$. Then, W_A has order 2. Let $w \in W_A$ be a non-trivial element of W_A , and m_w a representative of w in M' . The action of W_A on \widehat{M} is defined by

$$(w\sigma)(m) := \sigma(m_w^{-1}mm_w), \quad m \in M, \sigma \in \widehat{M}.$$

We have already associated the Selberg and Ruelle zeta functions with irreducible representations σ of M . These representations are chosen precisely to be the representations arising from restrictions of representations of K . Let $i^* : R(K) \rightarrow R(M)$ be the pullback of the embedding $i : M \hookrightarrow K$. We will distinguish the following two cases:

- **case (a):** σ is invariant under the action of the restricted Weyl group W_A .
- **case (b):** σ is not invariant under the action of the restricted Weyl group W_A .

In case (b), we define the following twisted zeta functions.

Definition 3.3. Let $\chi : \Gamma \rightarrow \operatorname{GL}(V_\chi)$ be a finite dimensional representation of Γ and $\sigma \in \widehat{M}$. The symmetrized zeta function $Z(s; \sigma, \chi)$ for X is defined by

$$S(s; \sigma, \chi) := Z(s; \sigma, \chi)Z(ws; \sigma, \chi), \quad (3.3)$$

where w is a non-trivial element of the restricted Weyl group W_A .

Definition 3.4. Let $\chi : \Gamma \rightarrow \operatorname{GL}(V_\chi)$ be a finite dimensional representation of Γ and $\sigma \in \widehat{M}$. The super zeta function $Z^s(s; \sigma, \chi)$ for X is defined by

$$Z^s(s; \sigma, \chi) := \frac{Z(s; \sigma, \chi)}{Z(ws; \sigma, \chi)}, \quad (3.4)$$

where w is a non-trivial element of the restricted Weyl group W_A .

Definition 3.5. Let $\chi : \Gamma \rightarrow \operatorname{GL}(V_\chi)$ be a finite dimensional representation of Γ and $\sigma \in \widehat{M}$. The super Ruelle zeta function $R^s(s; \sigma, \chi)$ for X is defined by

$$R^s(s; \sigma, \chi) := \frac{R(s; \sigma, \chi)}{R(s; w\sigma, \chi)}, \quad (3.5)$$

where w is a non-trivial element of the restricted Weyl group W_A .

We compute here the logarithmic derivative of the symmetrized and super zeta function.

Lemma 3.6. *Let*

$$L_{sym}(\gamma; \sigma) := \frac{\text{tr}(\chi(\gamma) \otimes \sigma(m_\gamma)) e^{-|\rho|l(\gamma)}}{\det(\text{Id} - \text{Ad}(m_\gamma a_\gamma)_{\bar{n}})}. \quad (3.6)$$

Then we have

1. *The logarithmic derivative of the symmetrized zeta function $S(s; \sigma, \chi)$ is given by*

$$L_S(s) := \frac{d}{ds} \log(S(s; \sigma, \chi)) = \sum_{[\gamma] \neq e} \frac{l(\gamma)}{n_\Gamma(\gamma)} L_{sym}(\gamma; \sigma + w\sigma) e^{-sl(\gamma)}. \quad (3.7)$$

2. *The logarithmic derivative of the super zeta function $Z^s(s; \sigma, \chi)$ is given by*

$$L^s(s) := \frac{d}{ds} \log(Z^s(s; \sigma, \chi)) = \sum_{[\gamma] \neq e} \frac{l(\gamma)}{n_\Gamma(\gamma)} L_{sym}(\gamma; \sigma - w\sigma) e^{-sl(\gamma)}. \quad (3.8)$$

Proof. 1. In case (b), for the symmetrized zeta function $S(s; \sigma, \chi)$, we see by equation (3.3)

$$\begin{aligned} \frac{d}{ds} \log(S(s; \sigma, \chi)) &= \frac{d}{ds} \log(Z(s; \sigma, \chi)) + \frac{d}{ds} \log(Z(s; w\sigma, \chi)) \\ &= \sum_{[\gamma] \neq e} \frac{l(\gamma)}{n_\Gamma(\gamma)} \text{tr}(\chi(\gamma) \otimes \sigma(m_\gamma)) \frac{e^{-sl(\gamma)} e^{-|\rho|l(\gamma)}}{\det(1 - \text{Ad}(m_\gamma a_\gamma)_{\bar{n}})} \\ &\quad + \sum_{[\gamma] \neq e} \frac{l(\gamma)}{n_\Gamma(\gamma)} \text{tr}(\chi(\gamma) \otimes w\sigma(m_\gamma)) \frac{e^{-sl(\gamma)} e^{-|\rho|l(\gamma)}}{\det(1 - \text{Ad}(m_\gamma a_\gamma)_{\bar{n}})} \\ &= \sum_{[\gamma] \neq e} \frac{l(\gamma)}{n_\Gamma(\gamma)} L_{sym}(\gamma; \sigma + w\sigma) e^{-sl(\gamma)}. \end{aligned}$$

2. In case (b), for the super zeta function $Z^s(s; \sigma, \chi)$, we see by equation (3.4)

$$\begin{aligned}
\frac{d}{ds} \log(Z^s(s; \sigma, \chi)) &= \frac{d}{ds} \log(Z(s; \sigma, \chi)) - \frac{d}{ds} \log(Z(s; w\sigma, \chi)) \\
&= \sum_{[\gamma] \neq e} \frac{l(\gamma)}{n_\Gamma(\gamma)} \operatorname{tr}(\chi(\gamma) \otimes \sigma(m_\gamma)) \frac{e^{-sl(\gamma)} e^{|\rho|l(\gamma)}}{\det(1 - \operatorname{Ad}(m_\gamma a_\gamma)_{\bar{\mathfrak{n}}})} \\
&\quad - \sum_{[\gamma] \neq e} \frac{l(\gamma)}{n_\Gamma(\gamma)} \operatorname{tr}(\chi(\gamma) \otimes w\sigma(m_\gamma)) \frac{e^{-sl(\gamma)} e^{|\rho|l(\gamma)}}{\det(1 - \operatorname{Ad}(m_\gamma a_\gamma)_{\bar{\mathfrak{n}}})} \\
&= \sum_{[\gamma] \neq e} \frac{l(\gamma)}{n_\Gamma(\gamma)} L_{\text{sym}}(\gamma; \sigma - w\sigma) e^{-sl(\gamma)}.
\end{aligned}$$

□

4 The twisted Dirac operator

Let $\sigma \in \widehat{M}$ be an irreducible representation of M with highest weight ν_σ as in (2.5). We recall that ν_n denotes the last coordinate of ν_σ . Let s be the spin representation of K . Since $d-1$ is an even integer, s splits into two irreducible half-spin representations (s^+, S^+) , (s^-, S^-) of M . Let $\operatorname{Cl}(\mathfrak{p})$ be the Clifford algebra of \mathfrak{p} with respect to the inner product $\langle \cdot, \cdot \rangle_0$, as in (2.1), restricted to \mathfrak{p} . Let $\cdot : \mathfrak{p} \otimes S \rightarrow S$ be the Clifford multiplication on $\mathfrak{p} \otimes S$. Let H be in \mathfrak{a}^+ , where \mathfrak{a}^+ is the Lie algebra of A^+ and A^+ is defined as in (2.2). Since M centralizes \mathfrak{a} , the Clifford multiplication by H preserves the decomposition $S = S^+ \oplus S^-$. The Clifford multiplication by H^2 acts as $-\operatorname{Id}$. Then, H acts on S^\pm with eigenvalues $\pm i$. Hence, we can consider the Clifford multiplication by H as multiplication by $\pm i \operatorname{sign}(\nu_n)$.

We consider the connection ∇ in $\operatorname{Cl}(\mathfrak{p})$, induced by the canonical connection in the tangent frame bundle of X . Let L be any bundle of left modules over $\operatorname{Cl}(\mathfrak{p})$ over \tilde{X} , i.e., a spinor bundle over \tilde{X} . We lift the connection ∇ in L and obtain a connection also denoted by ∇ . The Dirac operator $D : C^\infty(X, L) \rightarrow C^\infty(X, L)$ is defined as

$$D : C^\infty(X, L) \xrightarrow{\nabla} C^\infty(X, T^*X \otimes L) \xrightarrow{g} C^\infty(X, TX \otimes L) \xrightarrow{\cdot} C^\infty(X, L),$$

where we identify $TX \cong T^*X$ using the riemannian metric, and \cdot denotes the Clifford multiplication as above. Locally, it can be described as

$$Df \equiv \sum_{i=1}^d e_i \cdot \nabla_{e_i} f,$$

where (e_1, \dots, e_d) is a local orthonormal frame for $T_x X, x \in X$. The bundle L is a Dirac bundle over \tilde{X} . This means that

- the Clifford multiplication by unit vectors in $\text{Cl}(\mathfrak{p})$ is orthogonal i.e.,

$$\langle ef_1, ef_2 \rangle = \langle f_1, f_2 \rangle,$$

for all unit vectors $e \in T_x \tilde{X}, x \in \tilde{X}$ and all $f_1, f_2 \in L_x$, where L_x denotes the fiber of L over $x \in \tilde{X}$.

- the connection ∇ satisfies the product rule

$$\nabla(\phi f) = (\nabla \phi) \cdot f + \phi \cdot (\nabla f),$$

for all $\phi \in C^\infty(X, \text{Cl}(\mathfrak{p}))$ and all $f \in C^\infty(X, L)$.

The operator D is an elliptic ([LM89, Lemma 5.1]), formally self-adjoint ([LM89, Proposition 5.3]) operator of first order. We want to define twisted Dirac operators acting on smooth sections of vector bundles associated with the representations σ of M and arbitrary representations χ of Γ .

Proposition 4.1. *Let $\sigma \in \widehat{M}$. Then, there exists a unique element $\tau(\sigma) \in \widehat{K}$ and a splitting*

$$s \otimes \tau(\sigma) = \tau^+(\sigma) \oplus \tau^-(\sigma)$$

where $\tau^+(\sigma), \tau^-(\sigma) \in R(K)$ such that

$$\sigma + w\sigma = i^*(\tau^+(\sigma) - \tau^-(\sigma)) \tag{4.1}$$

Proof. This is proved in [BO95, Proposition 1.1, (3)]. □

We define the representation $\tau_s(\sigma)$ of K by

$$\tau_s(\sigma) := s \otimes \tau(\sigma), \tag{4.2}$$

with representation space $V_{\tau_s(\sigma)} = S \otimes V_{\tau(\sigma)}$, where $V_{\tau(\sigma)}$ is the representation space of $\tau(\sigma)$.

We consider the homogeneous vector bundle $\tilde{E}_{\tau(\sigma)}$ over \tilde{X} given by

$$\tilde{E}_{\tau(\sigma)} = G \times_{\tau(\sigma)} V_{\tau(\sigma)} \rightarrow \tilde{X}.$$

The vector bundle $\tilde{E}_{\tau_s(\sigma)} := \tilde{E}_{\tau(\sigma)} \otimes S$ over \tilde{X} carries a connection $\nabla^{\tau_s(\sigma)}$, defined by the formula

$$\nabla^{\tau_s(\sigma)} = \nabla^{\tau(\sigma)} \otimes 1 + 1 \otimes \nabla.$$

where $\nabla^{\tau(\sigma)}$ denotes the canonical connection in $\tilde{E}_{\tau(\sigma)}$. We extend the Clifford multiplication by requiring that it acts on $V_{\tau_s(\sigma)} = S \otimes V_{\tau(\sigma)}$ as follows.

$$e \cdot (\phi \otimes \psi) = (e \cdot \phi) \otimes \psi, \quad e \in \text{Cl}(\mathfrak{p}), \phi \in S, \psi \in V_{\tau(\sigma)}.$$

We define the Dirac operator $\tilde{D}(\sigma)$ acting on $C^\infty(\tilde{X}, V_{\tau_s(\sigma)})$ by

$$\tilde{D}(\sigma)f = \sum_{i=1}^d e_i \cdot \nabla_{e_i}^{\tau_s(\sigma)} f,$$

where (e_1, \dots, e_d) is local orthonormal frame for $T_x \tilde{X}$ and $f \in C^\infty(\tilde{X}, V_{\tau_s(\sigma)})$. The space of smooth sections $C^\infty(\tilde{X}, V_{\tau_s(\sigma)})$ can be identified with $C^\infty(G; \tau_s(\sigma))$ as in [Spi15, equation (5.1)].

Let now $\chi : \Gamma \rightarrow \text{GL}(V_\chi)$ be an arbitrary finite dimensional representation of Γ . Let E_χ be the associated flat vector bundle over X . Let $E_{\tau_s(\sigma)} := \Gamma \backslash \tilde{E}_{\tau_s(\sigma)}$ be the locally homogeneous vector bundle over X . We consider the product vector bundle $E_{\tau_s(\sigma)} \otimes E_\chi$ over X and we equip this bundle with the product connection $\nabla^{E_{\tau_s(\sigma)} \otimes E_\chi}$ defined by

$$\nabla^{E_{\tau_s(\sigma)} \otimes E_\chi} = \nabla^{E_{\tau_s(\sigma)}} \otimes 1 + 1 \otimes \nabla^{E_\chi}.$$

We consider the Clifford multiplication on $(V_{\tau_s(\sigma)} \otimes V_\chi)$ by requiring that it acts only on $V_{\tau_s(\sigma)}$, i.e.,

$$e \cdot (w \otimes v) = (e \cdot w) \otimes v, \quad e \in \text{Cl}(\mathfrak{p}), w \in V_{\tau_s(\sigma)}, v \in V_\chi.$$

For our proposal, we introduce the twisted Dirac operator $D_\chi^\sharp(\sigma)$ associated with $\nabla^{E_{\tau_s(\sigma)} \otimes E_\chi}$. We want to describe it locally. We consider an open subset of X such that $E_\chi|_U$ is trivial. Let $(v_j), j = 1, \dots, m$, be a basis of flat sections of $E_\chi|_U$, where $m = \text{rank}(E_\chi)$, and $\phi_j \in C^\infty(U, E_{\tau_s(\sigma)}|_U)$. Then,

$$E_{\tau_s(\sigma)} \otimes E_\chi|_U \cong \bigoplus_{j=1}^m E_{\tau_s(\sigma)}|_U,$$

and for each $\phi \in C^\infty(U, E_{\tau_s(\sigma)} \otimes E_\chi|_U)$,

$$\phi = \sum_{j=1}^m \phi_j \otimes v_j.$$

The product connection is given by

$$\nabla^{E_{\tau_s(\sigma)} \otimes E_\chi}(\phi) = \sum_{j=1}^m (\nabla^{E_{\tau_s(\sigma)}})(\phi_j) \otimes v_j.$$

Then the Dirac operator is described as follows.

$$\begin{aligned} D_\chi^\sharp(\sigma)\phi &= \sum_{i=1}^d e_i \cdot \nabla_{e_i}^{E_{\tau_s(\sigma)} \otimes E_\chi}(\phi) \\ &= \sum_{i=1}^d e_i \cdot \left(\sum_{j=1}^m (\nabla_{e_i}^{E_{\tau_s(\sigma)}})(\phi_j) \otimes v_j \right) \\ &= \sum_{i=1}^d \sum_{j=1}^m e_i \cdot ((\nabla_{e_i}^{E_{\tau_s(\sigma)}})(\phi_j) \otimes v_j). \end{aligned} \tag{4.3}$$

We consider the pullbacks $\tilde{E}_{\tau_s(\sigma)}, \tilde{E}_\chi$ to \tilde{X} of $E_{\tau_s(\sigma)}, E_\chi$, respectively, then, $\tilde{E}_\chi \cong \tilde{X} \times V_\chi$. We have

$$C(\tilde{X}, \tilde{E}_{\tau_s(\sigma)} \otimes \tilde{E}_\chi) \cong C(\tilde{X}, \tilde{E}_{\tau_s(\sigma)}) \otimes V_\chi.$$

With respect to this isomorphism, it follows from (4.3) that the lift $\tilde{D}_\chi^\sharp(\sigma)$ of the twisted Dirac operator $D_\chi^\sharp(\sigma)$ to \tilde{X} is of the form

$$\tilde{D}_\chi^\sharp(\sigma) = \tilde{D}(\sigma) \otimes \text{Id}_{V_\chi}. \tag{4.4}$$

We recall the definition of the operator $A_\chi^\sharp(\sigma)$ acting on $C^\infty(X, E(\sigma) \otimes E_\chi)$ from [Spi15, equation 5.26]

$$A_\chi^\sharp(\sigma) := \bigoplus_{m_\tau(\sigma) \neq 0} A_{\tau, \chi}^\sharp + c(\sigma),$$

where

$$c(\sigma) := -|\rho|^2 - |\rho_m|^2 + |\nu_\sigma + \rho_m|^2,$$

and ρ, ρ_m are defined by (2.3) and (2.4), respectively. The operator $A_{\tau, \chi}^\sharp$ is induced by the twisted Bochner-Laplace operator $\Delta_{\tau, \chi}^\sharp$ and acts on smooth sections of the twisted vector bundle $E(\sigma) \otimes E_\chi$ over X (see [Spi15, p. 27-28]). We recall here the definition of the vector bundle $E(\sigma)$. We consider always representations σ of M coming from restrictions of representations of K . Let $\tau_\sigma \in R(K)$ with $\tau_\sigma := \tau^+(\sigma) - \tau^-(\sigma)$. By

[BO95, Proposition 1.1], there exists unique integers $m_\tau(\sigma) \in \{-1, 0, 1\}$, which are equal to zero except for finitely many $\tau \in \widehat{K}$, such that for the case (b) (see p. 8)

$$\sigma + w\sigma = \sum_{\tau \in \widehat{K}} m_\tau(\sigma) i^*(\tau). \quad (4.5)$$

Then, the locally homogeneous vector bundle $E(\sigma)$ associated with τ is of the form

$$E(\sigma) = \bigoplus_{\substack{\tau \in \widehat{K} \\ m_\tau(\sigma) \neq 0}} E_\tau,$$

where E_τ is the locally homogeneous vector bundle associated with $\tau \in \widehat{K}$. Therefore, the vector bundle $E(\sigma)$ has a grading $E(\sigma) = E(\sigma)^+ \oplus E(\sigma)^-$. This grading is defined exactly by the positive or negative sign of $m_\tau(\sigma)$. Let $\widetilde{E}(\sigma)$ be the pullback of $E(\sigma)$ to \widetilde{X} . Then,

$$\widetilde{E}(\sigma) = \bigoplus_{\substack{\tau \in \widehat{K} \\ m_\tau(\sigma) \neq 0}} \widetilde{E}_\tau.$$

One can easily observe that by Proposition 4.1 (see also [BO95, Proposition 5.1 and p. 27-28]), that up to a \mathbb{Z}_2 grading, one can identify the two vector bundles $E(\sigma)$ and $E_{\tau_s(\sigma)}$.

We consider the lift $\widetilde{A}_\chi^\sharp(\sigma)$ of $A_\chi^\sharp(\sigma)$ to the universal covering \widetilde{X} . Then,

$$\begin{aligned} \widetilde{A}_\chi^\sharp(\sigma) &= \bigoplus_{m_\tau(\sigma) \neq 0} \widetilde{A}_{\tau, \chi}^\sharp + c(\sigma) \\ &= \bigoplus_{m_\tau(\sigma) \neq 0} (\widetilde{A}_\tau \otimes \text{Id}_{V_\chi}) + c(\sigma) \\ &= \bigoplus_{m_\tau(\sigma) \neq 0} (\widetilde{A}_\tau + c(\sigma)) \otimes \text{Id}_{V_\chi}. \end{aligned} \quad (4.6)$$

The Parthasarathy formula from [BO95, equation (1.11)] states

$$(\widetilde{D}(\sigma))^2 = \bigoplus_{m_\tau(\sigma) \neq 0} (\widetilde{A}_\tau + c(\sigma)) \quad (4.7)$$

If we combine (4.4), (4.6) and (4.7) the Parthasarathy formula generalizes as

$$(D_\chi^\sharp(\sigma))^2 = A_\chi^\sharp(\sigma). \quad (4.8)$$

5 Trace formulas

By (4.4), we get

$$(\tilde{D}_\chi^\sharp(\sigma))^2 = (\tilde{D}(\sigma))^2 \otimes \text{Id}_{V_\chi} \quad (5.1)$$

The square of the twisted Dirac operator $(D_\chi^\sharp(\sigma))^2$ acting on smooth sections of $E_{\tau_s(\sigma)} \otimes E_\chi$ is not a self-adjoint operator in general. Nevertheless, its principal symbol is given by

$$\sigma_{(D_\chi^\sharp(\sigma))^2}(x, \xi) = \|\xi\|^2 \otimes \text{Id}_{(V_{\tau_s(\sigma)} \otimes V_\chi)_x}, \quad x \in X, \quad \xi \in T_x^*X, \xi \neq 0.$$

Therefore, $(D_\chi^\sharp(\sigma))^2$ is a second order elliptic differential operator with nice spectral properties, i.e., its spectrum is discrete and contained in a translate of a positive cone $C \subset \mathbb{C}$. Furthermore, we can define the integral operators $e^{-t(D_\chi^\sharp(\sigma))^2}$ and $D_\chi^\sharp(\sigma)e^{-t(D_\chi^\sharp(\sigma))^2}$, and derive a corresponding trace formula. We introduce here a more general setting.

Setting 5.1. *Let $E \rightarrow X$ be a complex vector bundle over a smooth compact riemannian manifold X of dimension d . Let $D : C^\infty(X, E) \rightarrow C^\infty(X, E)$ be an elliptic differential operator of order $m \geq 1$. Let σ_D be its principal symbol.*

Definition 5.2. A spectral cut is a ray

$$R_\theta := \{\rho e^{i\theta} : \rho \in [0, \infty]\},$$

where $\theta \in [0, 2\pi)$.

Definition 5.3. The angle θ is a principal angle for an elliptic operator D if

$$\text{spec}(\sigma_D(x, \xi)) \cap R_\theta = \emptyset, \quad \forall x \in X, \forall \xi \in T_x^*X, \xi \neq 0.$$

Definition 5.4. We define the solid angle L_I associated with a closed interval I of \mathbb{R} by

$$L_I := \{\rho e^{i\theta} : \rho \in (0, \infty), \theta \in I\}.$$

Definition 5.5. The angle θ is an Agmon angle for an elliptic operator D , if it is a principal angle for D and there exists $\varepsilon > 0$ such that

$$\text{spec}(D) \cap L_{[\theta-\varepsilon, \theta+\varepsilon]} = \emptyset.$$

Lemma 5.6. *Let $\varepsilon \in (0, \frac{\pi}{2})$ be an angle such that the principal symbol $\sigma_D(x, \xi)$ of D , for $\xi \in T_x^*X, \xi \neq 0$ does not take values in $L_{[-\varepsilon, \varepsilon]}$. Then, the spectrum $\text{spec}(D)$ of the operator D is discrete and for every $\varepsilon \in (0, \frac{\pi}{2})$ there exist $R > 0$ such that $\text{spec}(D)$ is contained in the set $B(0, R) \cup L_{[-\varepsilon, \varepsilon]} \subset \mathbb{C}$.*

Proof. The discreteness of the spectrum follows from [Shu87, Theorem 8.4]. For the second statement see [Shu87, Theorem 9.3]. \square

Let λ_k be an eigenvalue of D and V_{λ_k} be the corresponding eigenspace. This is a finite dimensional subspace of $C^\infty(X, E)$ invariant under D . We have that for every $k \in \mathbb{N}$, there exist $N_k \in \mathbb{N}$ such that

$$(D - \lambda_k \text{Id})^{N_k} V_{\lambda_k} = 0$$

$$\lim_{k \rightarrow \infty} |\lambda_k| = \infty.$$

Definition 5.7. We call algebraic multiplicity $m(\lambda_k)$ of the eigenvalue λ_k the dimension of the corresponding eigenspace V_{λ_k} .

By Lemma 5.6, we get Lemma 5.8, which describes the spectrum of the square root $(D_\chi^\sharp(\sigma))^2$ of the twisted Dirac operator.

Lemma 5.8. *Let $\varepsilon \in (0, \frac{\pi}{2})$ be an angle such that the principal symbol $\sigma_{(D_\chi^\sharp(\sigma))^2}(x, \xi)$ of $(D_\chi^\sharp(\sigma))^2$, for $\xi \in T_x^*X, \xi \neq 0$ does not take values in $L_{[-\varepsilon, \varepsilon]}$. Then, the spectrum $\text{spec}((D_\chi^\sharp(\sigma))^2)$ of the twisted Dirac operator $(D_\chi^\sharp(\sigma))^2$ is discrete and for every ε there exists $R > 0$ such that $\text{spec}((D_\chi^\sharp(\sigma))^2)$ is contained in the set $B(-1, R) \cup L_{[-\varepsilon, \varepsilon]} \subset \mathbb{C}$.*

Proof. As in the proof of Lemma 5.6. \square

Let θ be an Agmon angle for the operator $(D_\chi^\sharp(\sigma))^2$. Then, by definition of the Agmon angle and Lemma 5.8, there exists $\varepsilon > 0$ such that

$$\text{spec}((D_\chi^\sharp(\sigma))^2) \cap L_{[\theta-\varepsilon, \theta+\varepsilon]} = \emptyset.$$

Since $(D_\chi^\sharp(\sigma))^2$ has discrete spectrum, there exists also an $r_0 > 0$ such that

$$\text{spec}((D_\chi^\sharp(\sigma))^2) \cap \{z \in \mathbb{C} : |z + 1| \leq 2r_0\} = \emptyset.$$

We define a contour Γ_{θ, r_0} as follows.

$$\Gamma_{\theta, r_0} = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3,$$

where $\Gamma_1 = \{-1 + re^{i\theta} : \infty > r \geq r_0\}$, $\Gamma_2 = \{-1 + r_0e^{ia} : \theta \leq a \leq \theta + 2\pi\}$, $\Gamma_3 = \{-1 + re^{i(\theta+2\pi)} : r_0 \leq r < \infty\}$. On Γ_1 , r runs from ∞ to r_0 , Γ_2 is oriented counterclockwise, and on Γ_3 , r runs from r_0 to ∞ . We put

$$e^{-t(D_\chi^\sharp(\sigma))^2} = \frac{i}{2\pi} \int_{\Gamma_{\theta, r_0}} e^{-t\lambda} ((D_\chi^\sharp(\sigma))^2 - \lambda \text{Id})^{-1} d\lambda \quad (5.2)$$

$$D_\chi^\sharp(\sigma) e^{-t(D_\chi^\sharp(\sigma))^2} = \frac{i}{2\pi} \int_{\Gamma_{\theta, r_0}} \lambda^{1/2} e^{-t\lambda} ((D_\chi^\sharp(\sigma))^2 - \lambda \text{Id})^{-1} d\lambda \quad (5.3)$$

We have $|e^{-t\lambda}| \leq e^{-t\operatorname{Re}(\lambda)}$. Furthermore, by [Shu87, Corollary 9.2], there exist a positive constant $c > 0$ such that $\|((D_\chi^\sharp(\sigma))^2 - \lambda \operatorname{Id})^{-1}\| \leq c|\lambda|^{-1}$. Hence, the integrals in (5.2) and (5.3) are well defined.

By [Mül11, Lemma 2.4], $D_\chi^\sharp(\sigma)e^{-t(D_\chi^\sharp(\sigma))^2}$ is an integral operator. Let $K_t^{\tau_s(\sigma), \chi}$ be its kernel function. Let F be a fundamental domain of Γ . We consider the space $L^2(\tilde{X}, \tilde{E}_{\tau_s(s)} \otimes \tilde{E}_\chi)^\Gamma$ of sections f of $\tilde{E}_{\tau_s(s)} \otimes \tilde{E}_\chi$ such that $f(\gamma\tilde{x}) = \gamma f(\tilde{x})$, $\forall \gamma \in \Gamma, \tilde{x} \in \tilde{X}$.

For $f \in L^2(X, E_{\tau_s(s)} \otimes E_\chi) \cong L^2(\tilde{X}, \tilde{E}_{\tau_s(s)} \otimes \tilde{E}_\chi)^\Gamma$, we have

$$\begin{aligned} D_\chi^\sharp(\sigma)e^{-t(D_\chi^\sharp(\sigma))^2}f(x) &= \int_X K_t^{\tau_s(\sigma), \chi}(x, y)f(y)dy \\ &= \int_{\tilde{X}} (K_t^{\tau_s(\sigma)}(\tilde{x}, \tilde{y}) \otimes \operatorname{Id}_{V_\chi})f(\tilde{y})d\tilde{y} \\ &= \sum_{\gamma \in \Gamma} \int_F (K_t^{\tau_s(\sigma)}(\tilde{x}, \gamma\tilde{y}) \otimes \chi(\gamma) \operatorname{Id}_{V_\chi})f(\tilde{y})d\tilde{y}, \end{aligned} \quad (5.4)$$

where $x, y \in X$ and $\tilde{x}, \tilde{y} \in \tilde{X}$ are lifts of x, y to \tilde{X} , respectively. The kernel function $K_t^{\tau_s(\sigma)}$ is the kernel associated with the operator $\tilde{D}(\sigma)e^{-t(\tilde{D}(\sigma))^2}$. It belongs to the Harish-Chandra L^q -Schwartz space $(\mathcal{C}^q(G) \otimes \operatorname{End}(V_{\tau_s(\sigma)}))^{K \times K}$, as it is defined in [BM83, p. 161-162]. Hence, we can interchange summation and integration in the right hand side of (5.4) and get

$$K_t^{\tau_s(\sigma), \chi}(x, x') = \sum_{\gamma \in \Gamma} K_t^{\tau_s(\sigma)}(g^{-1}\gamma g') \otimes \chi(\gamma),$$

where $x = \Gamma g, x' = \Gamma g', g, g' \in G$.

By [Mül11, Proposition 2.5], $D_\chi^\sharp(\sigma)e^{-t(D_\chi^\sharp(\sigma))^2}$ is a trace class operator, and its trace is given by

$$\operatorname{Tr}(D_\chi^\sharp(\sigma)e^{-t(D_\chi^\sharp(\sigma))^2}) = \sum_{\gamma \in \Gamma} \operatorname{tr} \chi(\gamma) \int_{\Gamma \backslash G} \operatorname{tr} K_t^{\tau_s(\sigma)}(g^{-1}\gamma g)dg.$$

We put

$$k_t^{\tau_s(\sigma)}(g) = \operatorname{tr} K_t^{\tau_s(\sigma)}(g). \quad (5.5)$$

We use the trace formula in [Mül11, Proposition 6.1] for non-unitary twists.

$$\begin{aligned} \operatorname{Tr}(D_\chi^\sharp(\sigma)e^{-t(D_\chi^\sharp(\sigma))^2}) &= \dim(V_\chi) \operatorname{Vol}(X)(k_t^{\tau_s(\sigma)})(e) \\ &\quad + \frac{1}{2\pi} \sum_{[\gamma] \neq e} \frac{l(\gamma) \operatorname{tr}(\chi(\gamma))}{n_\Gamma(\gamma)D(\gamma)} \sum_{\sigma \in \widehat{M}} \overline{\operatorname{tr} \sigma(m_\gamma)} \int_{\mathbb{R}} \Theta_{\sigma, \lambda}(k_t^{\tau_s(\sigma)})e^{-il(\gamma)\lambda}d\lambda. \end{aligned}$$

We continue analyzing the trace formula above in terms of characters. We want to compute the Fourier transform $\Theta_{\sigma,\lambda}(k_t^{\tau_s(\sigma)})$ of $k_t^{\tau_s(\sigma)}$. Following [MS89], we let (π, \mathcal{H}_π) be an unitary admissible representation of G in a Hilbert space \mathcal{H}_π . We let \mathcal{H}_π^∞ be the subspace of smooth vectors of \mathcal{H}_π . We set

$$\pi(K_t^{\tau_s(\sigma)}) := \int_G \pi(g) \otimes K_t^{\tau_s(\sigma)}(g) dg. \quad (5.6)$$

This defines a bounded trace class operator on $\mathcal{H}_\pi \otimes V_{\tau_s(\sigma)}$. By [BM83, p.160-161], relative to the splitting

$$\mathcal{H}_\pi \otimes V_{\tau_s(\sigma)} = (\mathcal{H}_\pi \otimes V_{\tau_s(\sigma)})^K \oplus [(\mathcal{H}_\pi \otimes V_{\tau_s(\sigma)})^K]^\perp,$$

$\tilde{\pi}(K_t^{\tau_s(\sigma)})$ has the form

$$\tilde{\pi}(K_t^{\tau_s(\sigma)}) = \begin{pmatrix} \pi(K_t^{\tau_s(\sigma)}) & 0 \\ 0 & 0 \end{pmatrix}, \quad (5.7)$$

with $\pi(K_t^{\tau_s(\sigma)})$ acting on $(\mathcal{H}_\pi \otimes V_{\tau_s(\sigma)})^K$. We consider orthonormal bases $(\xi_n), n \in \mathbb{N}, (e_j), j = 1, \dots, k$ of the vector spaces $\mathcal{H}_\pi, V_{\tau_s(\sigma)}$, respectively, where $k := \dim(V_{\tau_s(\sigma)})$. By (5.7), we have

$$\mathrm{Tr}(\pi(K_t^{\tau_s(\sigma)})) = \mathrm{Tr}(\tilde{\pi}(K_t^{\tau_s(\sigma)})). \quad (5.8)$$

Hence,

$$\begin{aligned} \mathrm{Tr}(\tilde{\pi}(K_t^{\tau_s(\sigma)})) &= \sum_n \sum_j \langle \tilde{\pi}(K_t^{\tau_s(\sigma)})(\xi_n \otimes e_j), (\xi_n \otimes e_j) \rangle \\ &= \sum_n \sum_j \int_G \langle \pi(g) \xi_n, \xi_n \rangle \langle K_t^{\tau_s(\sigma)}(g) e_j, e_j \rangle dg \\ &= \sum_n \int_G \langle \pi(g) \xi_n, \xi_n \rangle k_t^{\tau_s(\sigma)}(g) dg \\ &= \sum_n \langle \pi(k_t^{\tau_s(\sigma)}) \xi_n, \xi_n \rangle \\ &= \mathrm{Tr} \pi(k_t^{\tau_s(\sigma)}). \end{aligned} \quad (5.9)$$

Let $(X_i)_{i=1}^d$ be an orthonormal basis of \mathfrak{p} . We consider the operator acting on $(\mathcal{H}_\pi^\infty \otimes V_{\tau_s(\sigma)})^K$, defined by

$$\tilde{D}_{\tau_s(\sigma)}(\pi) := \sum_{i=1}^d X_i \cdot (\pi(X_i) \otimes \mathrm{Id}). \quad (5.10)$$

In [Pfa12, p.77], it is proved that $\tilde{D}_{\tau_s(\sigma)}(\pi)$ maps $(\mathcal{H}_\pi^\infty \otimes V_{\tau_s(\sigma)})^K$ to $(\mathcal{H}_\pi^\infty \otimes V_{\tau_s(\sigma)})^K$. By (5.6) we get

$$\tilde{\pi}(K_t^{\tau_s(\sigma)}) = e^{-tc(\sigma)} \tilde{D}_{\tau_s(\sigma)}(\pi) \circ \tilde{\pi}(H_t^{\tau_s(\sigma)}), \quad (5.11)$$

where

$$\tilde{\pi}(H_t^{\tau_s(\sigma)}) = \int_G \pi(g) \otimes H_t^{\tau_s(\sigma)} dg.$$

The kernel function $H_t^{\tau_s(\sigma)}$ corresponds to the integral operator

$$e^{-t(\tilde{D}(\sigma))^2} f(g) = e^{-tc(\sigma)} \int_G H_t^{\tau_s(\sigma)}(g^{-1}g') f(g') dg',$$

where $(\tilde{D}(\sigma))^2$ as in (4.7). $H_t^{\tau_s(\sigma)}$ belongs to the Harish-Chandra L^q -Schwartz space $(\mathcal{C}^q(G) \otimes \text{End}(V_{\tau_s(\sigma)}))^{K \times K}$.

As above (see equations (5.7), (5.8), (5.9)), the operator $\tilde{\pi}(H_t^{\tau_s(\sigma)})$ relative to the splitting,

$$\mathcal{H}_\pi \otimes V_{\tau_s(\sigma)} = (\mathcal{H}_\pi \otimes V_{\tau_s(\sigma)})^K \oplus [(\mathcal{H}_\pi \otimes V_{\tau_s(\sigma)})^K]^\perp,$$

takes the form

$$\tilde{\pi}(H_t^{\tau_s(\sigma)}) = \begin{pmatrix} \pi(H_t^{\tau_s(\sigma)}) & 0 \\ 0 & 0 \end{pmatrix}, \quad (5.12)$$

with $\pi(H_t^{\tau_s(\sigma)})$ acting on $(\mathcal{H}_\pi \otimes V_{\tau_s(\sigma)})^K$. Then, it follows that

$$e^{t\pi(\Omega)} \text{Id} = \pi(H_t^{\tau_s(\sigma)}), \quad (5.13)$$

where Id denotes the identity on the space $(\mathcal{H}_\pi^\infty \otimes V_{\tau_s(\sigma)})^K$ ([BM83, Corollary 2.2]). We have $(\mathcal{H}_\pi \otimes V_{\tau_s(\sigma)})^K = (\mathcal{H}_\pi^\infty \otimes V_{\tau_s(\sigma)})^K$ and

$$\text{Tr}(\pi(k_t^{\tau_s(\sigma)})) = e^{(\pi(\Omega) - c(\sigma))t} \text{Tr}(\tilde{D}_{\tau_s(\sigma)}(\pi)|_{(\mathcal{H}_\pi^\infty \otimes V_{\tau_s(\sigma)})^K}). \quad (5.14)$$

We recall that the representation space of $\tau_s(\sigma)$ is given by

$$V_{\tau_s(\sigma)} = V_{\tau(\sigma)} \otimes S.$$

Let π be the unitary principal series representation $\pi_{\sigma, \lambda}$ defined as in [Spi15, p. 7-8]. By [MS89, Proposition 3.6], we have for $(\sigma', V_{\sigma'}) \in \widehat{M}$,

$$\text{Tr}(\tilde{D}_{\tau_s(\sigma)}(\pi_{\sigma', \lambda})) = \lambda(\dim(V_{\sigma'} \otimes V_{\tau(\sigma)} \otimes S^+)^M - \dim(V_{\sigma'} \otimes V_{\tau(\sigma)} \otimes S^-)^M). \quad (5.15)$$

Following [Pfa12, Corollary 7.6], we let $\check{\sigma}'$ be the contragredient representation of σ' . Since $\check{\sigma}' \cong \sigma'$, we observe by equation (4.1) in Proposition 4.1 that for $\nu_n > 0$,

$$[\dim(V_{\sigma'} \otimes V_{\tau(\sigma)} \otimes S^+)^M - \dim(V_{\sigma'} \otimes V_{\tau(\sigma)} \otimes S^-)^M] = [\sigma - w\sigma : \sigma']. \quad (5.16)$$

Since $\sigma' \in \widehat{M}$ we have that $\sigma' \in \{\sigma, w\sigma\}$, otherwise the right hand side of (5.16) vanishes. In [Spi15, p. 28-29], the character $\Theta_{\sigma,\lambda}(q_t^\sigma)$ for the case (a), is computed. Recall that

$$\Theta_{\sigma,\lambda}(q_t^\sigma) = \sum_{\tau \in \widehat{K}} m_\tau(\sigma) \Theta_{\sigma,\lambda}(q_t^\tau),$$

where

$$q_t^\sigma = \sum_{\tau \in \widehat{K}} m_\tau(\sigma) q_t^\tau,$$

$$q_t^\tau = \text{tr } Q_t^\tau(g),$$

and $Q_t^\tau \in (\mathcal{C}^q(G) \otimes \text{End}(V_\tau))^{K \times K}$ is the kernel associated to the operator $e^{-t\tilde{A}_\tau}$. By [Spi15, equation (5.35)], we have

$$\Theta_{\sigma,\lambda}(q_t^\sigma) = \sum_{\tau \in \widehat{K}} m_\tau(\sigma) e^{t\pi_{\sigma,\lambda}(\Omega)} [\tau|_M : \sigma],$$

where

$$\pi_{\sigma,\lambda}(\Omega) = -\lambda^2 + c(\sigma).$$

This is proved in [Art, p.48]. We study here the case (b). This means that by (4.5), for $\sigma' \in \widehat{M}$,

$$\begin{aligned} \Theta_{\sigma',\lambda}(q_t^\sigma) &= e^{tc(\sigma)} e^{-t\lambda^2}, \quad \text{if } \sigma' \in \{\sigma, w\sigma\}, \\ \Theta_{\sigma',\lambda}(q_t^\sigma) &= 0, \quad \text{if } \sigma' \notin \{\sigma, w\sigma\}. \end{aligned} \quad (5.17)$$

If we put together (5.14)—(5.17), we obtain

$$\Theta_{\sigma',\lambda}(k_t^{\tau_s(\sigma)}) = \lambda e^{-t\lambda^2}, \quad \text{if } \sigma' = \sigma \quad (5.18)$$

$$\Theta_{\sigma',\lambda}(k_t^{\tau_s(\sigma)}) = -\lambda e^{-t\lambda^2}, \quad \text{if } \sigma' = w\sigma \quad (5.19)$$

$$\Theta_{\sigma',\lambda}(k_t^{\tau_s(\sigma)}) = 0, \quad \text{if } \sigma' \notin \{\sigma, w\sigma\}. \quad (5.20)$$

For the identity contribution we use the fact that when s is restricted to M it decomposes as $s^+ + s^-$. If $\nu_\sigma = (\nu_1, \dots, \nu_{n-1}, \nu_n)$ is the highest weight of σ , then the

highest weight of $w\sigma$ is given by $\nu_{w\sigma} = (\nu_1, \dots, \nu_{n-1}, -\nu_n)$. Specifically, for the half spin representations s^\pm we have

$$\nu_{ws^\pm} = \left(\frac{1}{2}, \dots, \mp \frac{1}{2}\right).$$

Hence,

$$ws^\pm = s^\mp.$$

The Plancherel polynomial is an even polynomial of λ and also $P_{s^+}(i\lambda) = P_{ws^+}(-i\lambda) = P_{s^-}(i\lambda)$. Hence,

$$\begin{aligned} k_t^{\tau_s(\sigma)}(e) &= \sum_{\sigma \in \widehat{M}} \int_{\mathbb{R}} \Theta_{\sigma, \lambda}(k_t^{\tau_s(\sigma)}) P_{\sigma}(i\lambda) d\lambda \\ &= \int_{\mathbb{R}} \lambda e^{-t\lambda^2} P_{s^+}(i\lambda) d\lambda + \int_{\mathbb{R}} -\lambda e^{-t\lambda^2} P_{s^-}(i\lambda) d\lambda = 0. \end{aligned} \quad (5.21)$$

For the hyperbolic contribution we use (5.18)—(5.20).

$$\mathrm{Tr}(D_{\chi}^{\sharp}(\sigma) e^{-t(D_{\chi}^{\sharp}(\sigma))^2}) = \frac{1}{2\pi} \sum_{[\gamma] \neq e} \frac{l(\gamma) \mathrm{tr}(\chi(\gamma) \otimes (\sigma(m_{\gamma}) - w\sigma(m_{\gamma})))}{D(\gamma) n_{\Gamma}(\gamma)} \int_{\mathbb{R}} \lambda e^{-t\lambda^2} e^{-il(\gamma)\lambda} d\lambda.$$

Equivalently,

$$\mathrm{Tr}(D_{\chi}^{\sharp}(\sigma) e^{-t(D_{\chi}^{\sharp}(\sigma))^2}) = \sum_{[\gamma] \neq e} \frac{-2\pi i}{(4\pi t)^{3/2}} \frac{l^2(\gamma) \mathrm{tr}(\chi(\gamma) \otimes (\sigma(m_{\gamma}) - w\sigma(m_{\gamma})))}{n_{\Gamma}(\gamma) D(\gamma)} e^{-l^2(\gamma)/4t}.$$

All in all, we have proved the following theorems.

Theorem 5.9. *For every $\sigma \in \widehat{M}$ we have for case (b)*

$$\mathrm{Tr}(D_{\chi}^{\sharp}(\sigma) e^{-t(D_{\chi}^{\sharp}(\sigma))^2}) = \sum_{[\gamma] \neq e} \frac{-2\pi i}{(4\pi t)^{3/2}} \frac{l^2(\gamma) \mathrm{tr}(\chi(\gamma) \otimes (\sigma(m_{\gamma}) - w\sigma(m_{\gamma})))}{n_{\Gamma}(\gamma) D(\gamma)} e^{-l^2(\gamma)/4t}. \quad (5.22)$$

Theorem 5.10. *For every $\sigma \in \widehat{M}$ we have for case (b)*

$$\begin{aligned} \mathrm{Tr}(e^{-tA_{\chi}^{\sharp}(\sigma)}) &= 2 \dim(V_{\chi}) \mathrm{Vol}(X) \int_{\mathbb{R}} e^{-t\lambda^2} P_{\sigma}(i\lambda) d\lambda \\ &\quad + \sum_{[\gamma] \neq e} \frac{l(\gamma)}{n_{\Gamma}(\gamma)} L_{sym}(\gamma; \sigma + w\sigma) \frac{e^{-l(\gamma)^2/4t}}{(4\pi t)^{1/2}}, \end{aligned} \quad (5.23)$$

where

$$L_{sym}(\gamma; \sigma) = \frac{\mathrm{tr}(\sigma(m_{\gamma}) \otimes \chi(\gamma)) e^{-|\rho|l(\gamma)}}{\det(\mathrm{Id} - \mathrm{Ad}(m_{\gamma} a_{\gamma})_{\bar{n}})}.$$

6 Meromorphic continuation of the super zeta function

Let $N \in \mathbb{N}$. Let $s_i, i = 1, \dots, N$ be complex numbers such that $s_i \in \mathbb{C} - \text{spec}(-D_\chi^\sharp(\sigma)^2)$. We consider the resolvent operator

$$R(s_i^2) = (D_\chi^\sharp(\sigma)^2 + s_i^2)^{-1}.$$

We want to obtain the trace class property of the operators

$$\prod_{i=1}^N R(s_i^2) \\ D_\chi^\sharp(\sigma) \prod_{i=1}^N R(s_i^2).$$

In order to obtain this property, we take sufficient large $N \in \mathbb{N}$, such that

- for $N > \frac{d}{2}$,

$$\text{Tr}\left(\prod_{i=1}^N R(s_i^2)\right) < \infty. \quad (6.1)$$

- for $N > \frac{d}{2} + 1$,

$$\text{Tr}(D_\chi^\sharp(\sigma) \prod_{i=1}^N R(s_i^2)) < \infty. \quad (6.2)$$

We denote the space of pseudodifferential operators of order k by ψDO^k . To prove the trace class property of the operators above, we observe at first that $\prod_{i=1}^N R(s_i^2) \in \psi DO^{-2N}$.

Let Δ be the Bochner-Laplace operator with respect to some metric, acting on $C^\infty(X, E_{\tau_s(\sigma)} \otimes E_\chi)$. Then, Δ is a second-order elliptic differential operator, which is formally self-adjoint and non-negative, i.e., $\Delta \geq 0$. Then, by Weyl's law, we have that for $N > \frac{d}{2}$,

$$(\Delta + \text{Id})^{-N}$$

is a trace class operator. Moreover,

$$B := (\Delta + \text{Id})^N \prod_{i=1}^N R(s_i^2)$$

is ψDO of order zero. Hence, it defines a bounded operator in $L^2(X, E_{\tau_s(\sigma)} \otimes E_\chi)$. Thus,

$$\prod_{i=1}^N R(s_i^2) = (\Delta + \text{Id})^{-N} B$$

is a trace class operator.

We recall here the following expressions of the resolvents. Let $s_1, \dots, s_N \in \mathbb{C}$ such that $\text{Re}(s_i^2) > -c$, for all $i = 1, \dots, N$, where c is a real number such that $\text{spec}(D_\chi^\sharp(\sigma)^2) \subset \{z \in \mathbb{C} : \text{Re}(z) > c\}$.

Then,

$$D_\chi^\sharp(\sigma)(D_\chi^\sharp(\sigma)^2 + s_i^2)^{-1} = \int_0^\infty e^{-ts_i^2} D_\chi^\sharp(\sigma) e^{-tD_\chi^\sharp(\sigma)^2} dt \quad (6.3)$$

$$(A_\chi^\sharp(\sigma) + s_i^2)^{-1} = \int_0^\infty e^{-ts_i^2} e^{-tA_\chi^\sharp(\sigma)} dt. \quad (6.4)$$

Proposition 6.1. *Let $N \in \mathbb{N}$ with $N > d/2 + 1$. Let $s_1, \dots, s_N \in \mathbb{C}$ with $s_i \neq s_j$ for all $i \neq j$ such that $\text{Re}(s_i^2) > -c$, for all $i = 1, \dots, N$, where c is a real number such that $\text{spec}(D_\chi^\sharp(\sigma)^2) \subset \{z \in \mathbb{C} : \text{Re}(z) > c\}$. Let $L^s(s) := \frac{d}{ds} \log(Z^s(s; \sigma, \chi))$ be the logarithmic derivative of the super zeta function. Then,*

$$\text{Tr}(D_\chi^\sharp(\sigma) \prod_{i=1}^N (D_\chi^\sharp(\sigma)^2 + s_i^2)^{-1}) = -\frac{i}{2} \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) L^s(s_i). \quad (6.5)$$

Proof. By [Spi15, Lemma 6.1] and formula (6.3), we have

$$D_\chi^\sharp(\sigma) \prod_{i=1}^N (D_\chi^\sharp(\sigma)^2 + s_i^2)^{-1} = \int_0^\infty \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) e^{-ts_i^2} D_\chi^\sharp(\sigma) e^{-tD_\chi^\sharp(\sigma)^2} dt.$$

The operators $D_\chi^\sharp(\sigma) \prod_{i=1}^N (D_\chi^\sharp(\sigma)^2 + s_i^2)^{-1}$, and $D_\chi^\sharp(\sigma) e^{-tD_\chi^\sharp(\sigma)^2}$ are both of trace class. Then,

$$\begin{aligned} D_\chi^\sharp(\sigma) \prod_{i=1}^N (D_\chi^\sharp(\sigma)^2 + s_i^2)^{-1} &= \int_\epsilon^\infty \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) e^{-ts_i^2} D_\chi^\sharp(\sigma) e^{-tD_\chi^\sharp(\sigma)^2} dt \\ &\xrightarrow[\epsilon \rightarrow 0]{R \rightarrow \infty} \int_\epsilon^R \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) e^{-ts_i^2} D_\chi^\sharp(\sigma) e^{-tD_\chi^\sharp(\sigma)^2} dt, \end{aligned}$$

where the limit is taken with respect to the trace norm $\|\mathcal{A}\|_1 := \text{Tr}|\mathcal{A}|$, with $\mathcal{A} = D_\chi^\sharp(\sigma) \prod_{i=1}^N (D_\chi^\sharp(\sigma)^2 + s_i^2)^{-1}$, or $D_\chi^\sharp(\sigma) e^{-tD_\chi^\sharp(\sigma)^2}$. We have

$$\begin{aligned} \text{Tr} \left(\int_0^\infty \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) e^{-ts_i^2} D_\chi^\sharp(\sigma) e^{-tD_\chi^\sharp(\sigma)^2} dt \right) = \\ \text{Tr} \left(\lim_{\substack{\epsilon \rightarrow 0 \\ R \rightarrow \infty}} \int_\epsilon^R \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) e^{-ts_i^2} D_\chi^\sharp(\sigma) e^{-tD_\chi^\sharp(\sigma)^2} dt \right). \end{aligned}$$

But,

$$\begin{aligned} \text{Tr} \left(\int_\epsilon^R \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) e^{-ts_i^2} D_\chi^\sharp(\sigma) e^{-tD_\chi^\sharp(\sigma)^2} dt \right) = \\ \int_\epsilon^R \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) e^{-ts_i^2} \text{Tr}(D_\chi^\sharp(\sigma) e^{-tD_\chi^\sharp(\sigma)^2}) dt. \end{aligned}$$

Hence, it is sufficient to show that the limit

$$\lim_{\substack{\epsilon \rightarrow 0 \\ R \rightarrow \infty}} \int_\epsilon^R \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) e^{-ts_i^2} \text{Tr}(D_\chi^\sharp(\sigma) e^{-tD_\chi^\sharp(\sigma)^2}) dt$$

exists. We study the behavior of the integral in the equation above as $\epsilon \rightarrow 0$.

By [Spi15, Lemma 6.3], we have that as $t \rightarrow 0^+$

$$\sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) e^{-ts_i^2} = O(t^{N-1}).$$

Also, by [GS95, Theorem 2.7, p.503-504], there exists a short time asymptotic expansion of the kernel of the operator $D_\chi^\sharp(\sigma) e^{-tD_\chi^\sharp(\sigma)^2}$

$$\text{Tr}(D_\chi^\sharp(\sigma) e^{-tD_\chi^\sharp(\sigma)^2}) \sim_{t \rightarrow 0^+} t^{-d/2}.$$

We have that as $t \rightarrow 0^+$

$$\left| \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) e^{-ts_i^2} \text{Tr}(D_\chi^\sharp(\sigma) e^{-tD_\chi^\sharp(\sigma)^2}) \right| \leq Ct,$$

where C is a positive constant. All in all, we have proved

$$\mathrm{Tr}(D_\chi^\sharp(\sigma) \prod_{i=1}^N (D_\chi^\sharp(\sigma)^2 + s_i^2)^{-1}) = \int_0^\infty \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) e^{-ts_i^2} \mathrm{Tr}(D_\chi^\sharp(\sigma) e^{-tD_\chi^\sharp(\sigma)^2}) dt.$$

We apply now the trace formula (5.22) for the operator $D_\chi^\sharp(\sigma) e^{-tD_\chi^\sharp(\sigma)^2}$. Then, we get

$$\begin{aligned} \mathrm{Tr}(D_\chi^\sharp(\sigma) \prod_{i=1}^N (D_\chi^\sharp(\sigma)^2 + s_i^2)^{-1}) &= \int_0^\infty \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) e^{-ts_i^2} \\ &\quad \left\{ \sum_{[\gamma] \neq e} \frac{-2\pi i}{(4\pi t)^{3/2}} \frac{l^2(\gamma) \mathrm{tr}(\chi(\gamma) \otimes (\sigma(m_\gamma) - w\sigma(m_\gamma)))}{n_\Gamma(\gamma) D(\gamma)} e^{-l^2(\gamma)/4t} \right\}. \end{aligned} \quad (6.6)$$

If we use the formula (see [EMOT54, p.146, (28)])

$$\int_0^\infty e^{-ts^2} \frac{1}{(4\pi t)^{3/2}} e^{-l^2(\gamma)/4t} dt = \frac{e^{-l(\gamma)s}}{4\pi l(\gamma)}$$

equation (6.6) becomes

$$\begin{aligned} \mathrm{Tr}(D_\chi^\sharp(\sigma) \prod_{i=1}^N (D_\chi^\sharp(\sigma)^2 + s_i^2)^{-1}) &= \frac{-i}{2} \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) \\ &\quad \left\{ \sum_{[\gamma] \neq e} \frac{l(\gamma) \mathrm{tr}(\chi(\gamma) \otimes (\sigma(m_\gamma) - w\sigma(m_\gamma)))}{n_\Gamma(\gamma) D(\gamma)} e^{-l(\gamma)s_i} \right\}. \end{aligned}$$

Hence, by equation (3.8) we get

$$\mathrm{Tr}(D_\chi^\sharp(\sigma) \prod_{i=1}^N (D_\chi^\sharp(\sigma)^2 + s_i^2)^{-1}) = \frac{-i}{2} \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) L^s(s_i).$$

□

The meromorphic continuation of the super zeta function follows from the Proposition (6.1) above.

Theorem 6.2. *The super zeta function $Z^s(s; \sigma, \chi)$ admits a meromorphic continuation to the whole complex plane \mathbb{C} . The singularities are located at $\{s_k^\pm = \pm i\lambda_k : \lambda_k \in \text{spec}(D_\chi^\sharp(\sigma)), k \in \mathbb{N}\}$ of order $\pm m_s(\lambda_k)$, where $m_s(\lambda_k) = m(\lambda_k) - m(-\lambda_k) \in \mathbb{N}$ and $m(\pm\lambda_k)$ denotes the algebraic multiplicity of the eigenvalue $\pm\lambda_k$.*

Proof. We define the function $\Phi(s_1, s_2, \dots, s_N)$ of the complex variables s_1, s_2, \dots, s_N by

$$\Phi(s_1, s_2, \dots, s_N) = -\frac{i}{2} \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) L^s(s_i). \quad (6.7)$$

By Lidskii's theorem and [Spi15, Lemma 6.1], (6.5) becomes

$$\sum_{\lambda_k} \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) m_s(\lambda_k) \lambda_k \frac{1}{(\lambda_k)^2 + s_i^2} = \Phi(s_1, s_2, \dots, s_N). \quad (6.8)$$

We fix the complex numbers $s_i, i = 2, \dots, N$ with $s_i \neq s_j$ for $i, j = 2, \dots, N$ and let the complex number $s = s_1$ vary. Hence,

$$\Phi(s, s_2, \dots, s_N) = \Phi(s)$$

The term that contains the logarithmic derivative $L^s(s)$ in $\Phi(s)$ is of the form

$$-\frac{i}{2} \left(\prod_{j=2}^N \frac{1}{s_j^2 - s^2} \right) L^s(s). \quad (6.9)$$

The term of

$$\sum_{\lambda_k} \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) m_s(\lambda_k) \lambda_k \frac{1}{(\lambda_k)^2 + s_i^2}$$

which is singular at $s = \pm i\lambda_k, k \in \mathbb{N}$ is

$$\left(\prod_{j=2}^N \frac{1}{s_j^2 - s^2} \right) m_s(\lambda_k) \lambda_k \frac{1}{(\lambda_k)^2 + s^2}.$$

If we multiply both sides of (6.8) by

$$2i \prod_{j=2}^N (s_j^2 - s^2),$$

we see that the residue of the logarithmic derivative $L^s(s)$ at $\pm i\lambda_k$ is $\pm m_s(\lambda_k)$.

By (3.8), $L^s(s)$ decreases exponentially as $\operatorname{Re}(s) \rightarrow \infty$. Hence, the integral

$$\int_s^\infty L^s(w)dw$$

over a path connecting s and infinity is well defined and

$$\log Z^s(s; \sigma, \chi) = - \int_s^\infty L^s(w)dw. \quad (6.10)$$

The integral above depends on the choice of the path, because $L^s(s)$ has singularities at s_k^\pm . Nevertheless, since all the residues of the singularities are integers, it follows that the exponential of the integral in the right hand side of (6.10) is independent of the choice of the path. The meromorphic continuation of the super zeta function $Z^s(s; \sigma, \chi)$ to the whole complex plane follows. \square

7 Meromorphic continuation of the symmetrized zeta function

Let $N \in \mathbb{N}$ with $N > d/2$. We choose $s_1, \dots, s_N \in \mathbb{C}$ with $s_i \neq s_j$ for all $i \neq j$ such that $\operatorname{Re}(s_i^2) > -r$, for all $i = 1, \dots, N$, where r is a real number such that $\operatorname{spec}(A_\chi^\sharp(\sigma)) \subset \{z \in \mathbb{C} : \operatorname{Re}(z) > r\}$.

Then, by [Spi15, Lemma 6.1] and equation (6.4), we have

$$\prod_{i=1}^N (A_\chi^\sharp(\sigma) + s_i^2)^{-1} = \int_0^\infty \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) e^{-ts_i^2} e^{-tA_\chi^\sharp(\sigma)} dt.$$

As in the proof of Proposition 6.1, we can consider the trace of the operators in the formula above and get

$$\operatorname{Tr} \prod_{i=1}^N (A_\chi^\sharp(\sigma) + s_i^2)^{-1} = \int_0^\infty \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) e^{-ts_i^2} \operatorname{Tr} e^{-tA_\chi^\sharp(\sigma)} dt.$$

We insert the trace formula (5.23) for the operator $e^{-tA_\chi^\sharp(\sigma)}$ and get

$$\begin{aligned} \text{Tr} \prod_{i=1}^N (A_{\tau, \chi}^\sharp(\sigma) + s_i^2)^{-1} = \\ \int_0^\infty \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) e^{-ts_i^2} \left\{ 2 \dim(V_\chi) \text{Vol}(X) \int_{\mathbb{R}} e^{-t\lambda^2} P_\sigma(i\lambda) d\lambda \right. \\ \left. + \sum_{[\gamma] \neq e} \frac{l(\gamma)}{n_\Gamma(\gamma)} L_{\text{sym}}(\gamma; \sigma + w\sigma) \frac{e^{-l(\gamma)^2/4t}}{(4\pi t)^{1/2}} \right\} dt. \end{aligned} \quad (7.1)$$

The first sum in the right hand side of (7.1) includes the double integral

$$I = \int_0^\infty \int_{\mathbb{R}} \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) e^{-ts_i^2} e^{-t\lambda^2} P_\sigma(i\lambda) d\lambda dt,$$

which has been computed in [Spi15, p. 33-34]

$$I = \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) \frac{\pi}{s_i} P_\sigma(s_i).$$

Hence, equation (7.1) reads

$$\begin{aligned} \text{Tr} \prod_{i=1}^N (A_\chi^\sharp(\sigma) + s_i^2)^{-1} = \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) \frac{\pi}{s_i} 2 \dim(V_\chi) \text{Vol}(X) P_\sigma(s_i) \\ + \sum_{i=1}^N \frac{1}{2s_i} \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) \sum_{[\gamma] \neq [e]} \frac{l(\gamma)}{n_\Gamma(\gamma)} L_{\text{sym}}(\gamma; \sigma + w\sigma) e^{-s_i l(\gamma)}. \end{aligned}$$

By (3.7), we can insert the logarithmic derivative $L_S(s)$ of the symmetrized zeta function. Then, we get

$$\begin{aligned} \text{Tr} \prod_{i=1}^N (A_\chi^\sharp(\sigma) + s_i^2)^{-1} = \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) \frac{\pi}{s_i} 2 \dim(V_\chi) \text{Vol}(X) P_\sigma(s_i) \\ + \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) \frac{1}{2s_i} L_S(s_i). \end{aligned} \quad (7.2)$$

Equation (7.2) will give the meromorphic continuation of the symmetrized zeta function.

Theorem 7.1. *The symmetrized zeta function $S(s; \sigma, \chi)$ admits a meromorphic continuation to the whole complex plane \mathbb{C} . The set of the singularities equals $\{s_k^\pm = \pm i\sqrt{\mu_k} : \mu_k \in \text{spec}(A_\chi^\sharp(\sigma)), k \in \mathbb{N}\}$. The orders of the singularities are equal to $m(\mu_k)$, where $m(\mu_k) \in \mathbb{N}$ denotes the algebraic multiplicity of the eigenvalue μ_k . For $\mu_0 = 0$, the order of the singularity s_0 is equal to $2m(0)$.*

Proof. By [Spi15, Lemma 6.1] and equation (7.2), we get

$$\begin{aligned} \sum_{\mu_k} \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) \frac{m(\mu_k)}{\mu_k + s_i^2} &= \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) \frac{\pi}{s_i} 2 \dim(V_\chi) \text{Vol}(X) P_\sigma(s_i) \\ &+ \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) \frac{1}{2s_i} L_S(s_i). \end{aligned}$$

We multiply the last equation by $2s_1$.

$$\begin{aligned} \sum_{\mu_k} \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) 2s_1 \frac{m(\mu_k)}{\mu_k + s_i^2} &= \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) \frac{4\pi s_1}{s_i} \dim(V_\chi) \text{Vol}(X) P_\sigma(s_i) \\ &+ \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) \frac{s_1}{s_i} L_S(s_i). \end{aligned} \quad (7.3)$$

We define the function $\Xi(s_1, \dots, s_N)$ of the complex variables s_1, \dots, s_N by

$$\Xi(s_1, \dots, s_N) := \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) \frac{s_1}{s_i} L_S(s_i).$$

We fix the complex numbers $s_i, i = 2, \dots, N$ with $s_i \neq s_j$ for $i, j = 2, \dots, N$ and let the complex number $s = s_1$ vary. Then,

$$\Xi(s, \dots, s_N) = \Xi(s),$$

and equation (7.3) becomes

$$\begin{aligned} \sum_{\mu_k} \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) 2s \frac{m(\mu_k)}{\mu_k + s_i^2} &= \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2} \right) \frac{4\pi s}{s_i} \dim(V_\chi) \text{Vol}(X) P_\sigma(s_i) \\ &+ \Xi(s). \end{aligned} \quad (7.4)$$

The term that contains the logarithmic derivative $L_S(s)$ in $\Xi(s)$ is of the form

$$\left(\prod_{j=2}^N \frac{1}{s_j^2 - s^2}\right) L_S(s). \quad (7.5)$$

The term of

$$\sum_{\mu_k} \sum_{i=1}^N \left(\prod_{\substack{j=1 \\ j \neq i}}^N \frac{1}{s_j^2 - s_i^2}\right) 2s \frac{m(\mu_k)}{\mu_k + s_i^2},$$

which is singular at $s = \pm i\sqrt{\mu_k}$, $k \in \mathbb{N}$ is

$$\left(\prod_{j=2}^N \frac{1}{s_j^2 - s^2}\right) 2s \frac{m(\mu_k)}{\mu_k + s^2}.$$

We multiply both sides of the equality (7.4) by

$$\prod_{j=2}^N (s_j^2 - s^2).$$

Then, the residues of $L_S(s)$ at the points $\pm i\sqrt{\mu_k}$ are $m(\mu_k)$, for $k \neq 0$ and $2m(0)$, for $k = 0$.

By (3.7), $L_S(s)$ decreases exponentially as $\operatorname{Re}(s) \rightarrow \infty$. Therefore, the integral

$$\int_s^\infty L_S(w) dw$$

over a path connecting s and infinity is well defined and

$$\log S(s; \sigma, \chi) = - \int_s^\infty L_S(w) dw. \quad (7.6)$$

The integral above depends on the choice of the path, because $L_S(s)$ has singularities. Since the residues of the singularities are integers, we can use the same argument as in the proof of Theorem 6.2. If we exponentiate the right hand side of (7.6), then this exponential is independent of the choice of the path. The meromorphic continuation of the symmetrized zeta function $S(s; \sigma, \chi)$ to the whole complex plane follows. \square

8 Meromorphic continuation of the Selberg and Ruelle zeta function

Theorem 8.1. *The Selberg zeta function $Z(s; \sigma, \chi)$ admits a meromorphic continuation to the whole complex plane \mathbb{C} . The set of the singularities equals to $\{s_k^\pm = \pm i\lambda_k : \lambda_k \in \text{spec}(D_\chi^\sharp(\sigma)), k \in \mathbb{N}\}$. The orders of the singularities are equal to $\frac{1}{2}(\pm m_s(\lambda_k) + m(\lambda_k^2))$. For $\lambda_0 = 0$, the order of the singularity is equal to $m(0)$.*

Proof. We observe at first that

$$Z(s; \sigma, \chi) = \sqrt{S(s; \sigma, \chi)Z^s(s; \sigma, \chi)}.$$

Recall that by equation (4.8) we have $A_\chi^\sharp(\sigma) = (D_\chi^\sharp(\sigma))^2$. Hence, we can identify the eigenvalues μ_k of $A_\chi^\sharp(\sigma)$ with λ_k^2 , where $\lambda_k \in \text{spec}(D_\chi^\sharp(\sigma))$. By Theorem 6.2 and Theorem 7.1, the product $S(s; \sigma, \chi)Z^s(s; \sigma, \chi)$ has its singularities at $s_k^\pm = \pm i\lambda_k$, of order $\pm m_s(\lambda_k) + m(\lambda_k^2)$. We need to prove that the order of the singularities of $Z(s; \sigma, \chi)$ is an even integer. This follows from the definition of the algebraic multiplicities $m_s(\lambda_k), m(\lambda_k^2)$ and the constuction of the locally homogenous vector bundles $E(\sigma), E_{\tau_s(\sigma)}$ associated to the representations $\tau(\equiv \tau_\sigma)$ and $\tau_s(\sigma)$ of K . By [Spi15, Proposition 5.4] together with equation (4.5), and Proposition 4.1 together with equation (4.2), we can choose representations τ of K , such that $E(\sigma) = E_{\tau_s(\sigma)}$ up to a \mathbb{Z}_2 -grading. Hence, $m_s(\lambda_k) \equiv m(\lambda_k^2) \pmod{2}$. The assertion follows. \square

Theorem 8.2. *For every $\sigma \in \widehat{M}$, the Ruelle zeta function $R(s; \sigma, \chi)$ admits a meromorphic continuation to the whole complex plane \mathbb{C} .*

Proof. The assertion follows from Theorem 8.1 together with [Spi15, Theorem 6.6]. \square

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